8.16 Cyclic Reduction

Let

$$S_1 = a_1 S_0 + b_1$$

$$S_i = a_i S_{i-1} + b_i \qquad i = 2, \ldots, n$$

One way to solve the recurrence in parallel is to use cyclic reduction:

From

$$S_i = a_i S_{i-1} + b_i$$

and

$$S_{i-1} = a_{i-1} S_{i-2} + b_{i-1}$$

we get

$$S_i = a_i a_{i-1} S_{i-2} + a_i b_{i-1} + b_i$$

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Which can be rewritten as

$$S_i = a_i^{(1)} S_{i-2} + b_i^{(1)}$$

where $a_i^{(1)}$ and $b_i^{(1)}$ are defined as follows:

$$a_i^{(1)} = a_i a_{i-1}$$

$$b_i^{(1)} = a_i b_{i-1} + b_i$$

Now we have S_i as a function of S_{i-2}.

If we repeat this process several times we obtain

$$S_i = a_i^{(l)} S_{i-2^l} + b_i^{(l)}$$

$$l = 0, 1, \ldots, \log n$$

$$i = 1, 2, \ldots, n$$

where

$$a_i^{(l)} = a_i^{(l-1)} a_{i-2^{l-1}}^{(l-1)}$$

$$b_i^{(l)} = a_i^{(l-1)} b_{i-2^{l-1}}^{(l-1)} + b_i^{(l-1)}$$

Initially,

$$a_i^{(0)} = a_i$$

$$b^{(0)} = b_i$$

When the subscript of a_i, b_i or S_i is outside the range $1,\ldots,n$, the value 0 should be assumed

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For the case n=8, the sequence of substitutions is as follows:

$$\begin{cases} S_1 = a_1 S_0 + b_1 \\ S_2 = a_2 S_1 + b_2 \\ S_3 = a_3 S_2 + b_3 \\ S_4 = a_4 S_3 + b_4 \\ S_5 = a_5 S_4 + b_5 \\ S_6 = a_6 S_5 + b_6 \\ S_7 = a_7 S_6 + b_7 \\ S_8 = a_8 S_7 + b_8 \end{cases} \rightarrow \begin{cases} S_1 = a_1 S_0 + b_1 \\ S_2 = a_2 a_1 S_0 + a_2 b_1 + b_2 \\ S_3 = a_3 a_2 S_1 + a_3 b_2 + b_3 \\ S_4 = a_4 a_3 S_2 + a_4 b_3 + b_4 \\ S_5 = a_5 a_4 S_3 + a_5 b_4 + b_5 \\ S_6 = a_6 a_5 S_4 + a_6 b_5 + b_6 \\ S_7 = a_7 a_6 S_5 + a_7 b_6 + b_7 \\ S_8 = a_8 a_7 S_6 + a_8 b_7 + b_8 \end{cases} \rightarrow \begin{cases} S_1 = a_1^{(1)} S_0 + b_1^{(1)} \\ S_2 = a_2^{(1)} S_0 + b_2^{(1)} \\ S_3 = a_3^{(1)} S_1 + b_3^{(1)} \\ S_4 = a_4^{(1)} S_2 + b_4^{(1)} \\ S_5 = a_5^{(1)} S_3 + b_5^{(1)} \\ S_6 = a_6^{(1)} S_4 + b_6^{(1)} \\ S_7 = a_7^{(1)} S_5 + b_7^{(1)} \\ S_8 = a_8^{(1)} S_6 + b_8^{(1)} \end{cases}$$

$$
\begin{cases}
S_1 = a_1^{(1)} S_0 + b_1^{(1)} \\
S_2 = a_2^{(1)} S_0 + b_2^{(1)} \\
S_3 = a_3^{(1)} S_1 + b_3^{(1)} \\
S_4 = a_4^{(1)} S_2 + b_4^{(1)} \\
S_5 = a_5^{(1)} S_3 + b_5^{(1)} \\
S_6 = a_6^{(1)} S_4 + b_6^{(1)} \\
S_7 = a_7^{(1)} S_5 + b_7^{(1)} \\
S_8 = a_8^{(1)} S_6 + b_8^{(1)}
\end{cases}
\rightarrow
\begin{cases}
S_1 = a_1^{(1)} S_0 + b_1^{(1)} \\
S_2 = a_2^{(1)} S_0 + b_2^{(1)} \\
S_3 = a_3^{(1)} a_1^{(1)} S_0 + a_3^{(1)} b_1^{(1)} + b_3^{(1)} \\
S_4 = a_4^{(1)} a_2^{(1)} S_0 + a_4^{(1)} b_2^{(1)} + b_4^{(1)} \\
S_5 = a_5^{(1)} a_3^{(1)} S_1 + a_5^{(1)} b_3^{(1)} + b_5^{(1)} \\
S_6 = a_6^{(1)} a_4^{(1)} S_2 + a_6^{(1)} b_4^{(1)} + b_6^{(1)} \\
S_7 = a_7^{(1)} a_5^{(1)} S_3 + a_7^{(1)} b_5^{(1)} + b_7^{(1)} \\
S_8 = a_8^{(1)} a_6^{(1)} S_4 + a_8^{(1)} b_6^{(1)} + b_8^{(1)}
\end{cases}
\rightarrow
\begin{cases}
S_1 = a_1^{(2)} S_0 + b_1^{(2)} \\
S_2 = a_2^{(2)} S_0 + b_2^{(2)} \\
S_3 = a_3^{(2)} S_0 + b_3^{(2)} \\
S_4 = a_4^{(2)} S_0 + b_4^{(2)} \\
S_5 = a_5^{(2)} S_1 + b_5^{(2)} \\
S_6 = a_6^{(2)} S_2 + b_6^{(2)} \\
S_7 = a_7^{(2)} S_3 + b_7^{(2)} \\
S_8 = a_8^{(2)} S_4 + b_8^{(2)}
\end{cases}
$$

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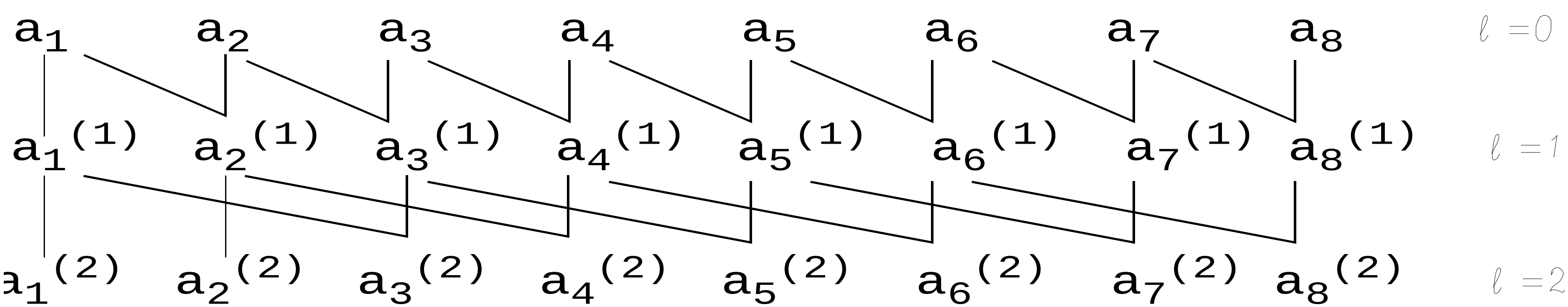
$$
\left\{\left\{\begin{array}{lcl}
S_1 & = & a_1^{(2)} S_0 + b_1^{(2)} \\
S_2 & = & a_2^{(2)} S_0 + b_2^{(2)} \\
S_3 & = & a_3^{(2)} S_0 + b_3^{(2)} \\
S_4 & = & a_4^{(2)} S_0 + b_4^{(2)} \\
S_5 & = & a_5^{(2)} S_1 + b_5^{(2)} \\
S_6 & = & a_6^{(2)} S_2 + b_6^{(2)} \\
S_7 & = & a_7^{(2)} S_3 + b_7^{(2)} \\
S_8 & = & a_8^{(2)} S_4 + b_8^{(2)}
\end{array}\right.\right.
\rightarrow
\left\{\begin{array}{lcl}
S_1 & = & a_1^{(2)} S_0 + b_1^{(2)} \\
S_2 & = & a_2^{(2)} S_0 + b_2^{(2)} \\
S_3 & = & a_3^{(2)} S_0 + b_3^{(2)} \\
S_4 & = & a_4^{(2)} S_0 + b_4^{(2)} \\
S_5 & = & a_5^{(2)} a_1^{(2)} S_0 + a_5^{(2)} b_1^{(2)} + b_5^{(2)} \\
S_6 & = & a_6^{(2)} a_2^{(2)} S_0 + a_6^{(2)} b_2^{(2)} + b_6^{(2)} \\
S_7 & = & a_7^{(2)} a_3^{(2)} S_0 + a_7^{(2)} b_3^{(2)} + b_7^{(2)} \\
S_8 & = & a_8^{(2)} a_4^{(2)} S_0 + a_8^{(2)} b_4^{(2)} + b_8^{12}
\end{array}\right.
\rightarrow
\left\{\begin{array}{lcl}
S_1 & = & a_1^{(3)} S_0 + b_1^{(3)} \\
S_2 & = & a_2^{(3)} S_0 + b_2^{(3)} \\
S_3 & = & a_3^{(3)} S_0 + b_3^{(3)} \\
S_4 & = & a_4^{(3)} S_0 + b_4^{(3)} \\
S_5 & = & a_5^{(3)} S_0 + b_5^{(3)} \\
S_6 & = & a_6^{(3)} S_0 + b_6^{(3)} \\
S_7 & = & a_7^{(3)} S_0 + b_7^{(3)} \\
S_8 & = & a_8^{(3)} S_0 + b_8^{(3)}
\end{array}\right.
$$

When $l = \log n$

$$S_i = a^{(\log n)} S_0 + b_i^{(\log n)}$$

To compute the *a*'s and *b*'s in parallel we proceed as shown below (only for $\ell = 1$ and 2 is shown.).



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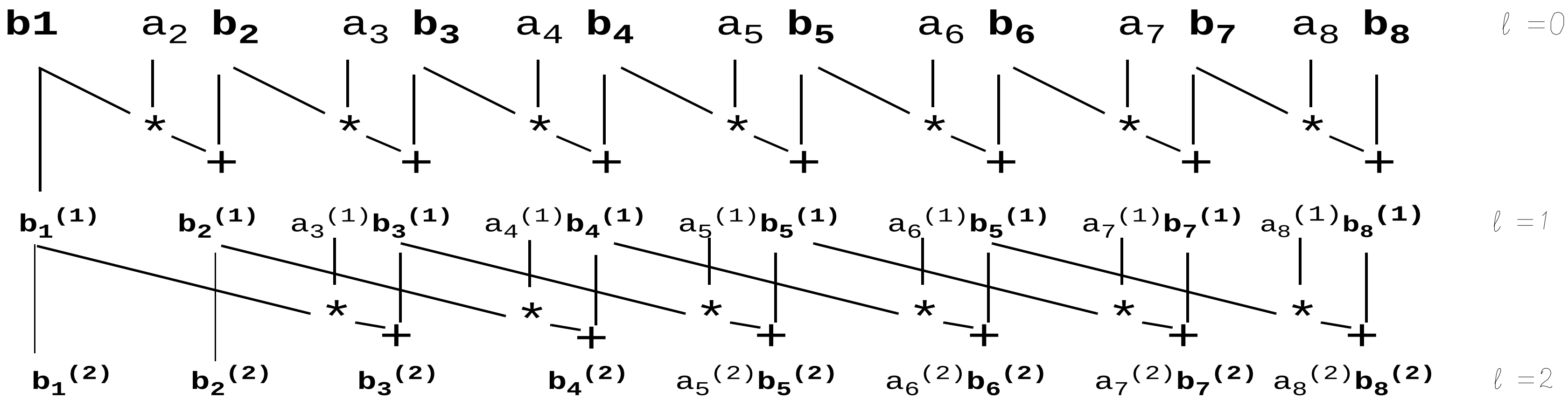

The resulting program in Fortran 90 is

```
S(1:n)=b(1:n)
do i=1,logn
   S(1:n)=EOSHIFT(S(1:n),-2**(i-
1))*a(1:n)+S(1:n)
   a(1:n)=EOSHIFT(a(1:n),-2**(i-1))*a(1:n)
end do
```

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